Decision Theory

Stat 510

WANT TO ESTIMATE θ WITH SOME $\hat{\theta}$.

MLE

MOM

POSTERIOR MEAN

HOW TO CHOOSE?

DECISION THEORY

$\hat{\theta}$ → ESTIMATOR

→ DECISION RULE

LOSS FUNCTION

MEASURES "DIFFERENCE" BETWEEN θ AND $\hat{\theta}$.

* $L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$ SQUARED ERROR LOSS

$L(\theta, \hat{\theta}) = |\theta - \hat{\theta}|$ ABSOLUTE LOSS

$L(\theta, \hat{\theta}) = \begin{cases} 0 & \theta = \hat{\theta} \\ 1 & \theta \neq \hat{\theta} \end{cases}$ ZERO-ONE LOSS

DEFN

THE RISK OF ESTIMATOR $\hat{\theta}$ IS

FIXED

$$R(\theta, \hat{\theta}) = \mathbb{E}_\theta\left[L(\theta, \hat{\theta})\right] = \int L(\theta, \hat{\theta})\, f(x; \theta)\, dx$$

DEPENDS ON DISTRIBUTION OF X

WITH SQUARED ERROR LOSS

$$R(\theta, \hat{\theta}) = \mathbb{E}\left[(\hat{\theta} - \theta)^2\right] = MSE = \mathbb{V}[\hat{\theta}] + (\text{BIAS}(\hat{\theta}))^2$$

Defn

Maximum Risk $\longrightarrow$ minimax

$$\bar{R}(\hat{\theta}) = \sup_{\theta} R(\theta, \hat{\theta})$$

Bayes Risk $\longrightarrow$ Bayes rule

prior for θ.

$$r(f, \hat{\theta}) = \int R(\theta, \hat{\theta})\, f(\theta)\, d\theta$$

Example

$$X_1, X_2, \dots, X_n \sim \text{Bernoulli}(p)$$

$$\hat{p}_1 = \frac{1}{n} \sum X_i = \bar{X}$$ ← MLE

$$\hat{p}_2 = \frac{\sum X_i + \alpha}{\alpha + \beta + n}$$ ← Posterior mean with Beta(α, β) prior

Risk? with squared error loss

Example

$$R(p, \hat{p}_1) = \mathbb{V}[\bar{X}] = \frac{p(1-p)}{n}$$

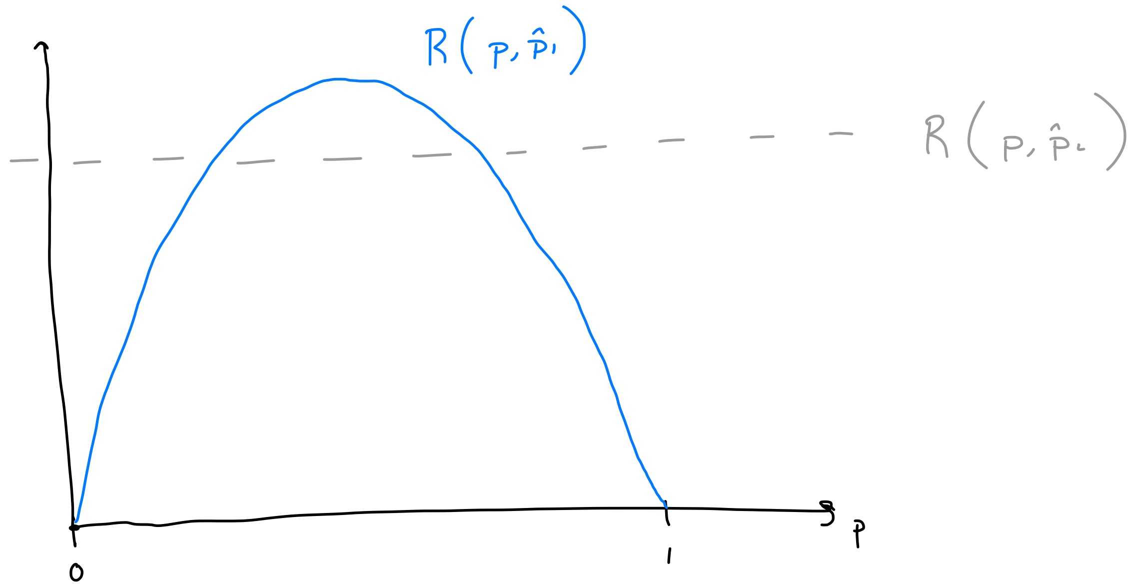
$$R(p, \hat{p}_2) = \mathbb{V}[\hat{p}_2] + \left(\text{BIAS}(\hat{p}_2)\right)^2$$

$$= \frac{np(1-p)}{(\alpha+\beta+n)^2} + \left(\frac{np+\alpha}{\alpha+\beta+n} - p\right)^2$$

WITH $\alpha = \beta = \sqrt{n/4}$

$$= \frac{n}{4(n+\sqrt{n})^2}$$

EXAMPLE



Example

$$\bar{R}(\hat{p}_1) = \max_{0<p<1} \frac{p(1-p)}{n} = \frac{1}{4n}$$

$$\bar{R}(\hat{p}_2) = \max_{p} \frac{n}{4(n+\sqrt{n})^2} = \frac{n}{4(n+\sqrt{n})^2}$$

"BETTER"

Example

$f(p) = 1$ prior

$$r(f, \hat{p}_1) = \int R(p, \hat{p}_1) f(p)\, dp = \frac{1}{6n}$$

"Better"

$n \geq 20$

$$r(f, \hat{p}_2) = \int R(p, \hat{p}_2) f(p)\, dp = \frac{n}{4(n+\sqrt{n})^2}$$

DEFN

An estimator $\hat{\theta}$ is <u>inadmissible</u> if there exists some $\hat{\theta}'$ such that

$$R(\theta, \hat{\theta}') \leq R(\theta, \hat{\theta}) \quad \text{for all } \theta$$

$$R(\theta, \hat{\theta}') < R(\theta, \hat{\theta}) \quad \text{for at least one } \theta$$

Otherwise, $\hat{\theta}$ is <u>admissible</u>

Stein's Paradox

$$\theta = (\theta_1, \theta_2, \ldots, \theta_k)$$

$$X = (X_1, X_2, \ldots, X_k) \qquad X_i \sim N(\theta_i, 1)$$

Loss $\quad L(\theta, \hat{\theta}) = \sum_{j=1}^{k} (\theta_j - \hat{\theta}_j)^2$

$$\hat{\theta}^{MLE} = (X_1, X_2, \ldots, X_k)$$

James-Stein $\quad \hat{\theta}^{JS} = (\hat{\theta}_1^{JS}, \ldots, \hat{\theta}_k^{JS})$

$$\hat{\theta}_i^{JS} = \left(1 - \frac{k-2}{\sum x_i^2}\right)^+ X_i$$

SHRINKS ESTIMATES TOWARDS 0

$$(z)^+ = \max(z, 0)$$

IF $k \geq 3$

$$R(\theta, \hat{\theta}^{JS}) < R(\theta, \hat{\theta}^{MLE})$$!!!