Parametric Estimator Properties

Stat 510

LAST TIME

HOW TO CONSTRUCT ESTIMATORS OF PARAMETERS?

- MoM
- MLE

Point Estimation

Parameter of interest

Use $\hat{\theta}$ as "best guess" for θ.

Fixed, unknown

Given $X_1, X_2, \ldots X_n \sim F$

Point estimator $\longrightarrow$ $\hat{\theta}_n = g(\underbrace{X_1, X_2, \ldots X_n}_{\text{Data}})$

Function

Random Variable

Sampling Distribution...

Method of Moments

Distribution is a function of the parameter

Parameter $\theta = (\theta_1, \dots \theta_k)$

j^{th} moment $\alpha_j \equiv \alpha_j(\theta) = E_\theta[X^j] = \int x^j \, dF_\theta(x)$

moments are functions of the parameter

j^{th} sample moment $\hat{\alpha}_j = \frac{1}{n} \sum_{i=1}^{n} X_i^j$

METHOD OF MOMENTS   MoM

THE MoM ESTIMATOR $\hat{\theta}_n$ IS THE VALUE OF θ SUCH THAT

$$\alpha_1(\hat{\theta}_n) = \hat{\alpha}_1$$

$$\alpha_2(\hat{\theta}_n) = \hat{\alpha}_2$$

SAMPLE SECOND MOMENT

SECOND MOMENT

$$\vdots$$

$$\alpha_k(\hat{\theta}_n) = \hat{\alpha}_k$$

k EQUATIONS, k UNKNOWNS $\longrightarrow$ SOLVE

THM

LET $\hat{\theta}$ BE THE MOM ESTIMATOR FOR θ. THEN UNDER APPROPRIATE CONDITIONS

- $\hat{\theta}$ EXISTS WITH PROBABILITY TENDING TOWARDS 1. EXISTENCE
- $\hat{\theta} \xrightarrow{P} \theta$ CONSISTENT
- $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \Sigma)$ ASYMPTOTIC NORMALITY

SPECIFICATION GIVEN IN BOOK

EASIER TO BOOTSTRAP

MLE PROPERTIES

- THE MLE IS CONSISTENT, $\hat{\theta} \xrightarrow{P} \theta$
- THE MLE IS EQUIVARIANT. IF $\hat{\theta}$ IS MLE OF θ THEN $g(\hat{\theta})$ IS MLE OF $g(\theta)$
- MLE IS ASYMPTOTICALLY NORMAL

$$\frac{\hat{\theta} - \theta}{\hat{se}} \xrightarrow{d} N(0,1)$$

- MLE IS EFFICIENT FOR LARGE SAMPLES
- MLE IS APPROX THE BAYES ESIMATOR.

ASSUMING REGULARITY CONDITIONS

Defn

Score Function

$$s(x;\theta) = \frac{d}{d\theta} \log f(x;\theta)$$

Fisher Information

$$I_n(\theta) = \mathbb{V}\left[\sum_{i=1}^{n} s(X_i;\theta)\right]$$

$$= \sum_{i=1}^{n} \mathbb{V}\left[s(X_i;\theta)\right]$$

THM

$n = 1 \qquad I(\theta) = I_1(\theta)$

$$I(\theta) = -\mathbb{E}\left[\frac{d^2}{d\theta^2} \log f(x;\theta)\right]$$

$$= -\int \left(\frac{d^2}{d\theta^2} \log f(x;\theta)\right) f(x;\theta)\, dx$$

AND

$$I_n(\theta) = n\, I(\theta)$$

FOR IID CASE

Thm LARGE SAMPLE NORMALITY OF MLE

LET $SE = \sqrt{\mathbb{V}[\hat{\theta}]}$

- $SE \approx \sqrt{1/I_n(\theta)}$ AND $\dfrac{\hat{\theta}_n - \theta}{SE} \xrightarrow{d} N(0,1)$

- $\hat{SE} \approx \sqrt{1/I_n(\hat{\theta})}$ AND $\dfrac{\hat{\theta}_n - \theta}{\hat{SE}} \xrightarrow{d} N(0,1)$

$$\hat{\theta} \approx N(\theta, \hat{SE})$$

THM

DEFINE

$$C_n = \left(\hat{\theta}_n - z_{\alpha/2}\, \hat{se},\ \hat{\theta}_n + z_{\alpha/2}\, \hat{se} \right)$$

THEN

$$P(\theta \in C_n) \longrightarrow 1-\alpha \quad \text{AS} \quad n \to \infty$$

"two"

SO

$$\hat{\theta}_n \pm 2\, \hat{se}$$

IS AN APPROX 95% CI

Example

$X_1, \ldots, X_n$ IID Bern(p)

MLE $\hat{p} = \frac{1}{n} \sum_{i=1}^{n} X_i$

$f(x;p) = p^x (1-p)^{x-1}$

$\log f(x;p) = x \log p + (x-1) \log(1-p)$

$$s(X;p) = \frac{d}{dp} \log f(x;p) = \frac{X}{p} - \frac{X-1}{1-p}$$

$$-s'(X;p) = \frac{X}{p^2} + \frac{1-X}{(1-p)^2}$$

$$I(p) = \mathbb{E}\left[-s'(X;p)\right] = \frac{p}{p^2} + \frac{1-p}{(1-p)^2} = \frac{1}{p(1-p)}$$

Continued Example

$$I_n(p) = n\,I(p) = \frac{n}{p(1-p)}$$

$$\hat{SE} = \sqrt{1/I_n(\hat{p})} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Approx 95% CI

$$\hat{p} \pm 2\sqrt{\frac{\hat{p}\,(1-\hat{p})}{n}}$$

CONSIDER ESTIMATORS T_n AND U_n FOR θ SUCH THAT

$$\sqrt{n}\,(T_n - \theta) \xrightarrow{d} N(0, t^2)$$

$$\sqrt{n}\,(U_n - \theta) \xrightarrow{d} N(0, u^2)$$

THEN DEFINE THE ASYMPTOTIC RELATIVE EFFICIENCY OF U TO T TO BE

$$ARE(U, T) = \frac{t^2}{u^2}$$

EXAMPLE

$X_1 \ldots X_n \sim N(\theta, \sigma^2)$

- $\hat{\theta} = \frac{1}{n} \sum x_i$ MLE

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2)$$

- $\tilde{\theta} =$ SAMPLE MEDIAN

$$\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N\left(0, \frac{\pi}{2}\sigma^2\right)$$

THEN

$$ARE(\tilde{\theta}, \hat{\theta}) = \frac{\sigma^2}{\frac{\pi}{2}\sigma^2} = 2/\pi = 0.63 < 1$$

Thm

$\hat{\theta}$ is MLE

$\tilde{\theta}$ is any other estimator

Then

$$\mathrm{ARE}\left(\tilde{\theta}, \overset{\text{MLE}}{\hat{\theta}}\right) \leq 1$$

THM DELTA METHOD $\hat{\theta}$ is MLE of θ

IF $\tau = g(\theta)$ AND $g'(\theta) \neq 0$ $\hat{\tau} = g(\hat{\theta})$

THEN

$$\frac{\hat{\tau} - \tau}{\hat{se}(\hat{\tau})} \xrightarrow{d} N(0,1)$$

$$\hat{se}(\hat{\tau}) = |g'(\hat{\theta})| \, \hat{se}(\hat{\theta})$$

$\hat{\tau} \pm 2\,\hat{se}(\hat{\tau})$ is APPROX 95% CI

EXAMPLE

$$X_1, \ldots, X_n \sim N(\mu, \sigma^2)$$

KNOWN

"SOME STUFF"

WANT TO ESTIMATE $\psi = \log \sigma$ WITH 95% CI

$$\log \mathcal{L}(\sigma) = -n \log \sigma - \frac{1}{2\sigma^2} \sum (x_i - \mu)^2 + ?$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2}$$

$$\log f(X; \sigma) = -\log \sigma - \frac{(X-\mu)^2}{2\sigma^2} + ?$$

$$\log f(x;\sigma) = -\log \sigma - \frac{(x-\mu)^2}{2\sigma^2} + ?$$

$$\frac{d}{d\sigma} \log f(x;\sigma) = \frac{-1}{\sigma} + \frac{(x-\mu)^2}{\sigma^3}$$

$$\frac{d^2}{d\sigma^2} \log f(x;\sigma) = \frac{1}{\sigma^2} - \frac{3(x-\mu)^2}{\sigma^4}$$

$$\mathbb{E}\left[(x-\mu)^2\right] = \mathbb{V}\left[x\right] = \sigma^2$$

$$I(\sigma) = -\mathbb{E}\left[\frac{1}{\sigma^2} - \frac{3(x-\mu)^2}{\sigma^4}\right] = -\frac{1}{\sigma^2} + \frac{3\sigma^2}{\sigma^4} = \frac{2}{\sigma^2}$$

$$I_n(\sigma) = n\,I(\sigma) = \frac{2n}{\sigma^2}$$

$$\hat{se}(\hat{\sigma}) = \sqrt{1/I_n(\sigma)} = \sigma/\sqrt{2n}$$

$$\psi = g(\sigma) = \log \sigma$$

$$g'(\sigma) = \frac{1}{\sigma}$$

$$\hat{se}(\hat{\psi}) = \left|\frac{1}{\hat{\sigma}}\right| \frac{\hat{\sigma}}{\sqrt{2n}} = \frac{1}{\sqrt{2n}}$$

$$\boxed{\hat{\psi} \pm 2 \cdot \frac{1}{\sqrt{2n}}}$$

Additional Items

- Information matrix
- Multiparameter delta method
- Sufficiency + Factorization Thm
- Rao-Blackwell
- Exponential Families