STAT 510: Homework 07

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Due: Monday, April 4, 11:59 PM

Exercise 1 (Poisson Fisher Information)

Let $X_1, X_2, \ldots, X_n \sim \text{Poisson}(\lambda)$. Find the method of moments estimator of λ. Find the maximum likelihood estimator of λ. Find the Fisher information $I(\lambda)$.

Exercise 2 (Fisher Information Matrix)

Let $X_1, X_2, \ldots, X_n \sim \text{N}(\mu, \sigma^2)$. Find $I_n(\mu, \sigma)$.

Exercise 3 (Exponential MLE)

Let $X_1, X_2, \ldots, X_n \sim \text{Exponential}(\lambda)$. That is

$$f(x) = \lambda e^{-\lambda x}.$$

Use the MLE and its standard error to derive an expression for an approximate 95% confidence interval for λ.

Exercise 4 (Exponential MLE, Continued)

Define $\phi = \log(\lambda)$. Use the MLE and its standard error to derive an expression for an approximate 95% confidence interval for ϕ.

```r
set.seed(42)
exp_data = rexp(n = 100, rate = 0.5)
```

Using the data stored in `exp_data`, calculate an approximate 95% confidence interval for λ two ways:

- Using the interval from Exercise 3.
- Using the interval from this exercise, transformed back to the λ scale.

Exercise 5 (Another MLE)

Let $X_1, X_2, \ldots, X_n \sim \text{N}(\theta, 1)$. Define

$$Y_i = \begin{cases} 1 & \text{if } X_i > 0 \\ 0 & \text{if } X_i \leq 0. \end{cases}$$

Let $\phi = \mathbb{P}(Y_1 = 1)$.

Use the MLE, $\hat{\phi}$, and its standard error to derive an expression for an approximate 95% confidence interval for ϕ.

Exercise 6 (Asymptotic Relative Efficiency)

Continuing the setup from Exercise 5, now define

$$\tilde{\phi} = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

Find the asymptotic relative efficiency of $\tilde{\phi}$ to $\hat{\phi}$. Your answer will be a function of θ. Provide the value of of θ and the associated asymptotic relative efficiency for the value of θ that gives the largest asymptotic relative efficiency.

Exercise 7 (Comparing Two Groups)

Suppose n_1 people are given treatment 1 and n_2 people are given treatment 2. Let X_1 be the number of people on treatment 1 who respond favorably to the treatment and let X_2 be the number of people on treatment 2 who respond favorably.

Assume $X_1 \sim \text{Binomial}(n_1, p_1)$ and $X_2 \sim \text{Binomial}(n_2, p_2)$.

Let $\phi = p_1 - p_2$.

Use the MLE, $\hat{\phi}$, and its standard error to derive an expression for an approximate 90% confidence interval for ϕ. To arrive at the standard error, first find $I(p_1, p_2)$ and then apply the delta method.

Exercise 8 (Comparing Standard Errors)

Continue with the setup from Exercise 7. Given:

- $n_1 = n_2 = 200$
- $X_1 = 160$
- $X_2 = 148$

Compare 90% confidence interval for ϕ using standard errors from Exercise 7 and the parametric bootstrap.

Exercise 9 (Geometric MLE)

Let $X_1, X_2, \ldots, X_n \sim \text{Geometric}(\pi)$.

Use the MLE, $\hat{\pi}$, and its standard error to derive an expression for an approximate 95% confidence interval for π.

Exercise 10 (Geometric MLE, Continued)

Define $\psi = \text{logit}(\pi)$. Use the MLE and its standard error to derive an expression for an approximate 95% confidence interval for ψ.

```r
set.seed(42)
geom_data = rgeom(n = 100, prob = 0.2)
```

Using the data stored in `geom_data`, calculate an approximate 95% confidence interval for π two ways:

- Using the interval from Exercise 9.
- Using the interval from this exercise, transformed back to the π scale.

```r
# shift data to match parameterization use in previous problem
geom_data = geom_data + 1
```

Exercise 11 (Rao-Blackwellization)

Let $X_1, X_2, \ldots, X_n \sim \text{Poisson}(\lambda)$. Show that $\sum_{i=1}^n X_i$ is a sufficient statistic for λ. Consider two estimators:

1. $\hat{\lambda}_1 = X_1$
2. $\hat{\lambda}_2$ which is the results of applying Rao-Blackwell to $\hat{\lambda}_1 = X_1$ with $\sum_{i=1}^n X_i$.

Show that $\hat{\lambda}_2$ has a smaller MSE than $\hat{\lambda}_1$.